Null output is obtained for three values of γ by adjusting A and ϕ. Using these data, (7), (13), and (16) are solved to obtain C_1-C_3. The rotating transition is returned to ports 5 and 6, and ports 3 and 4 are short circuited so that $\rho_{34}'=\rho_{43}'=0$. Again, A and ϕ are adjusted for null output at three values of γ. Using these data and values of C_1-C_3 previously determined, (7) and (11) through (15) are combined to yield C_4-C_6. The apparatus is now calibrated.

To measure Kerr effect, a sample is placed at ports 3 and 4, and the off-diagonal terms in (11) become

$$\left.\begin{aligned}\rho_{34}' &= \alpha - \beta\\ \rho_{43}' &= \alpha + \beta\end{aligned}\right\}. \tag{17}$$

With the rotating transition set at some arbitrary angle γ, A and ϕ are adjusted for null output with and without the applied magnetic field B. Inserting those values of A and ϕ into (7) and (13) yields two values of R_2. The single value R_3 is found from γ with (15). Equation (14) evaluated with and without the magnetic field thus becomes

$$R_3 = \frac{R_2(0) + \alpha}{\alpha R_2(0) + 1} \tag{18}$$

and

$$R_3 = \frac{R_2(B) + \{\alpha - \beta\}}{\{\alpha + \beta\} R_2(B) + 1}\cdot \tag{19}$$

The magnetic Kerr effect follows by solving (18) and (19) simultaneously for β. Although calculation of the preceding equations is straightforward, it is fairly time consuming. For this reason, we have programmed a digital computer to determine β directly from the calibration and measurement data obtained from the apparatus.

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On the Theory of Shielded Surface Waves

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Abstract—**An analysis is given for the modes which will be excited between two parallel impedance boundaries. It is shown that, for inductive-type surfaces, two of these modes have a surface wave character even though the structure is bounded in the transverse dimension. The interaction between these surface waves and the accompanying waveguide modes is discussed for this model which is admittedly highly idealized.**

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Introduction

In a recent interesting paper, Barlow [1] has considered the possibility that a surface wave may be shielded in such a manner that its transverse field pattern is limited. He proposed the application of the concept to communication circuits for high-speed trains. The main advantage would be its insensitivity to outside interference.

In order to provide further insight into the nature of shielded surface waves, a rather simple model is chosen here.

Consideration is given the TM modes which will propagate in the x direction in air between two parallel supporting surfaces (of infinite extent) in the xz plane. The relevant field components are H_z, E_x, and E_y which vary as $\exp[i\omega t]$.

The Mode Sum

The tangential field components are assumed to be related by surface impedances Z_0 and Z_d. Thus,

$$E_x/H_y = +Z_0 \qquad \text{at } y = 0$$

and

$$E_x/H_y = -Z_d \qquad \text{at } y = d. \tag{1}$$

Because we are dealing with a bounded structure, it is assumed the fields can be written as a discrete sum of modes. Thus, suppose that [2]

$$H_z = \sum_n a_n f_n(y) \exp[-i\lambda_n x] \tag{2}$$

where a_n is a coefficient, $f_n(y)$ is the transverse variation of the modes, and λ_n is the longitudinal wavenumber. Within the region $0<y<d$, H_z satisfies

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + k^2)H_z = 0 \tag{3}$$

where k is the wavenumber in air. From Maxwell's equations, $E_x=(i\epsilon\omega)^{-1}\partial H_z/\partial y$ and $E_y=-(i\epsilon\omega)^{-1}\partial H_z/\partial x$, where ϵ is the dielectric constant of free space.

An application of the boundary conditions shows that λ_n is to be determined from

$$\left(\frac{u_n - i\epsilon\omega Z_0}{u_n + i\epsilon\omega Z_0}\right)\left(\frac{u_n - i\epsilon\omega Z_d}{u_n + i\epsilon\omega Z_d}\right)\exp[-2u_n d] = 1 \tag{4}$$

where $u_n=(\lambda_n^2-k^2)^{1/2}$. Here,

$$f_n(y) = \frac{\exp[+u_n y] + R_0 \exp[-u_n y]}{2R_0^{1/2}} \tag{5}$$

is chosen where

$$R_0 = \frac{u_n - i\epsilon\omega Z_0}{u_n + i\epsilon\omega Z_0}\,.$$

In a straightforward manner, it may readily be shown that

$$N_{n,n'} = \frac{1}{d}\int_0^d f_n(y) f_{n'}(y)\,dy = 0 \qquad \text{for } n \neq n' \tag{6}$$

while

$$2N_{n,n} = 1 - \frac{i\epsilon\omega Z_0/d}{u^2 + (\epsilon\omega Z_0)^2} - \frac{i\epsilon\omega Z_d/d}{u^2 + (\epsilon\omega Z_d)^2}\,. \tag{7}$$

The orthogonality property exhibited by (6) assures that only a discrete spectrum is needed for this problem, as indicated by (2).

At the aperture plane $x=0$, assume that the electric field distribution $g(y)$ is known. Thus,

$$E_y\big|_{x=0} = E_0 g(y) \tag{8}$$

where E_0 is a constant with dimensions volts per meter. By utilizing (6) and (7), it is readily found that

$$a_n = \frac{E_0}{d}\,\frac{\epsilon\omega}{N_{n,n}\lambda_n}\int_0^d g(y) f_n(y)\,dy. \tag{9}$$

Inserting this into (2), we then obtain the formal exact solution of the problem. This formal development constitutes a modest extension of the well-known analysis for excitation of modes in a parallel-plate region with perfectly conducting walls [3], [4]. It is surprising that the explicit form, valid for surface impedance boundaries, is apparently not available in standard microwave engineering textbooks, although equivalent results are found in the radio propagation literature [2], [5].

The "Surface Wave" Modes

To facilitate the discussion, assume that both boundaries are purely inductive. Thus, $Z_0=i(k/\epsilon\omega)p$ and $Z_d=i(k/\epsilon\omega)q$, where p and q are dimensionless real quantities. The modal condition (4) is thus written in the form[1]

$$\left(\frac{u - kp}{u + kp}\right)\left(\frac{u - kq}{u + kq}\right) = \exp[-2ud]. \tag{10}$$

If the real part of ud is sufficiently large, the solutions are of the form $u\cong kp$ and $u\cong kq$. The corresponding values of the horizontal wavenumber are

$$\lambda = k(1 + p^2)^{1/2} \quad \text{and} \quad k(1 + q^2)^{1/2}.$$

These are "slow" surface waves as the phase velocities are $(1+p^2)^{-1/2}$ and $(1+q^2)^{1/2}$ relative to the plane wave velocity in the air. The transverse variation of these modes is described by

$$\frac{f(y)}{f(0)} \cong \exp[-kpy] \tag{11a}$$

and

$$\frac{f(y)}{f(0)} \cong \exp[-kq(d - y)], \tag{11b}$$

which indicate that the waves "cling" to the boundary surfaces. Within this approximation, the "surface waves" are noninteracting and their properties are identical to the trapped waves which may propagate on single flat surfaces.

[1] When referring to the surface wave modes, the subscript n is dropped.

When the interaction is not strong, it is desirable to consider another form of (10). This is written

$$(u - kp) = \delta(u) \tag{12}$$

where $\delta(u)=(u+kp)(u-kq)^{-1}(u+kq)\exp[-2ud]$, and where $\delta(u)$ is to be regarded as a perturbation. Thus, provided p is not near q, proceed to obtain a first-order perturbation by replacing u on the right-hand side of (12) by kp, the zero-order solution. The corrected solution for one of the surface waves is then given by

$$u \cong kp + \delta(kp) \tag{13}$$

where

$$\delta(kp) \cong 2kp\frac{p+q}{p-q}\exp[-2kpd]. \tag{14}$$

The interaction effect is exemplified by the finiteness of δ which vanishes if kpd is sufficiently large, provided $p \neq q$.

In a similar manner, the first-order corrected solution for the other surface wave is given by

$$u \cong kq + \delta(kq) \tag{15}$$

where

$$\delta(kq) \cong 2kq\frac{q+p}{q-p}\exp[-2kqd]. \tag{16}$$

In the special case $p=q$, the solutions are obtained by writing (10) in the form

$$(u - kp) = \pm\Delta(u) \tag{17}$$

where

$$\Delta(u) = (u + kp)\exp[-ud].$$

The corresponding first-order perturbation solutions are

$$u \cong kp \pm \Delta(kp) \tag{18}$$

where

$$\Delta(kp) \cong 2kp\exp[-kpd].$$

It is interesting to note that the solution given by (18) is double valued even though the two boundaries are identical. A closer inspection of the matter indicates that these give rise to even- and odd-type field patterns about the center line of the guide.

A case of some interest is when $p \neq q$, but $p-q$ is small. To handle this situation, we define a small parameter x by the relation

$$kq = kp + x.$$

Equation (10) then takes the form

$$(u - kp)^2 - x(u - kp) - \delta_x^2(u) = 0 \tag{19}$$

where

$$\delta_x^2(u) = [(u + kp)^2 + x(u + kp)]\exp[-2ud].$$

The first-order perturbation solution of (19) yields

$$(u - kp) \cong \frac{x}{2} \pm \left[\delta_x^2 + \left(\frac{x}{2}\right)^2\right]^{1/2} \tag{20}$$

where

$$\delta_x^2 \cong 4(kp)^2\exp[-2kpd].$$

As x tends to zero, this is seen to reduce to (18).

Once the value of u appropriate for a given situation has been obtained, the longitudinal wavenumber λ is obtained from the relationship $\lambda=(u^2+k^2)^{1/2}$. These are part of the discrete sum indicated by (2). It appears that the interaction does no more than modify the magnitude of u and λ so that, for a lossless structure, these parameters remain real and positive.

The Waveguide Modes

In addition to the surface wave contributions, an infinite number of waveguide modes may exist. For the lossless structure, they are characterized by solutions λ which are either purely real or purely imaginary. The latter correspond to the waveguide modes beyond cutoff and the modes are evanescent in the positive x direction.

To solve for the waveguide modes, it is convenient to introduce the dimensionless parameters C and S which are defined by

$$u = ikC \quad \text{and} \quad \lambda = kS.$$

The mode equation (10) may thus be written

$$\left(\frac{C-ip}{C+ip}\right)\left(\frac{C-iq}{C+iq}\right)\exp[-2ikCd] = \exp[-2\pi im] \tag{21}$$

where now m is considered to be zero or a positive integer. When both p and $q \ll |C|$, it follows that (21) is approximated by

$$kdC = \pi m - (p + q)C^{-1}. \tag{22}$$

This may be solved as a quadratic to yield

$$2C_m = \left(\frac{\pi m}{kd}\right) \pm \left[\left(\frac{\pi m}{kd}\right)^2 - \frac{4(p+q)}{kd}\right]^{1/2}. \tag{23}$$

The positive sign before the radical is taken, as C_m must reduce to $(\pi m/kd)$ for the case where $p=q=0$. Thus,

$$S_m = \left\{1 - \left(\frac{\pi m}{kd}\right)^2\frac{1}{4} \cdot\left[1 + \left(1 - \frac{4kd(p+q)}{(\pi m)^2}\right)^{1/2}\right]^2\right\}^{1/2}. \tag{24}$$

It is interesting to note that when $m=0$, (24) gives $S_0=[1+(p+q)/kd]^{1/2}$ which, however, is only valid if $p+q \ll kd$. Actually, this zero-order waveguide mode splits into the two surface wave modes when $(p+q)kd \gg 1$ as indicated in the previous section. Therefore, in what follows and in order to avoid confusion, assume that only the positive-order waveguide modes need be discussed. For $m=1, 2, 3, \cdots$, the radical in (24) is expanded, which is valid when $(p+q)kd \ll 1$. Then,

$$S_m \cong \left[1 - \left(\frac{\pi m}{kd}\right)^2 + \frac{2(p+q)}{kd}\right]^{1/2}$$

$$\cong -i\left[\left(\frac{\pi m}{kd}\right)^2 - 1 - \frac{2(p+q)}{kd}\right]^{1/2}. \quad (25)$$

The latter form is to be used when the modes are "cut off," (i.e., $\pi m/kd > 1$).

A convenient but approximate form of (26) is obtained by expanding the radical and retaining only the first two terms. Combining this with a similar result derived from (25), we find that

$$S_m \cong \left[1 - \left(\frac{\pi m}{kd}\right)^2\right]^{1/2} + \frac{(p+q)}{kd}\left[1 - \left(\frac{\pi m}{kd}\right)^2\right]^{-1/2} \quad (26)$$

where $m=1, 2, 3, \cdots$. It is evident that this result is only valid if

$$(p+q) \ll kd\left[1 - \left(\frac{\pi m}{kd}\right)^2\right]^{-1}$$

which restricts its usefulness to surface reactances small compared with 120π ohms. This limiting case is not of immediate concern and it would not be applicable under conditions where noninteracting surface waves are supported by the structure.

The function $f_m(y)$, characterizing the transverse field variation of the waveguide modes, is really identical to (5) but, in the present notation, it reads

$$f_m(y) = \frac{\exp[+ikC_m y] + R_0 \exp[-ikC_m y]}{2R_0^{1/2}} \quad (27)$$

where

$$R_0 = \frac{C_m - ip}{C_m + ip}. \quad (28)$$

For $p \ll C_m$, it is seen that

$$f_m(y) \cong \cos(kC_m y)$$

which demonstrates the sinusoidal character of the transverse variation for the waveguide modes.

The normalization function $N_{m,m}$ for the waveguide modes is also formally given by (7) but a more convenient form is

$$2N_{m,m} = 1 - \frac{p}{kd}\frac{1}{C_m^2 + p^2} - \frac{q}{kd}\frac{1}{C_m^2 + q^2}. \quad (29)$$

It is interesting to note that if $p \ll q$,

$$2N_{m,m} = 1 + \frac{\sin 2kdC_m}{2kdC_m} \quad (30)$$

where use has been made of the resonant condition given by (21). If both p and $q \ll 1$, it follows from either (29) or (30) that $N_{m,m} \cong \frac{1}{2}$ for $m=1, 2, 3, \cdots$.

Power Considerations

The power flow P_n, in the guide for a given mode, is calculated from

$$P_n = \text{Re}\int_0^d E_y H_z^* dy \text{ watts per unit width.} \quad (31)$$

Carrying out the integrations, we readily obtain the result that

$$P_n = \text{Re}\, \eta d |a_n|^2 S_n N_{n,n} \quad (32)$$

where $\eta = k/\epsilon\omega$ and where a_n is given by (9). If the aperture is a slit excited by a voltage V_0, it is seen that

$$a_n = (V_0/d)(\epsilon\omega/\lambda_n)N_{n,n}^{-1} f_n(y_s) \quad (33)$$

where y_s is the y coordinate of the slit. Then, if we are dealing with the lossless structure employed above, it follows that

$$P_m = V_0^2 \frac{f_m^2(y_s)}{\eta d S_m N_{m,m}} \text{ watts per unit width} \quad (34)$$

for the waveguide modes of order m.

We are now able to say that the ratio of the power P in a surface wave mode to the power P_m in a waveguide mode is very approximately given by

$$\frac{P}{P_m} \cong \frac{f^2(y_s)}{f_m^2(y_s)} \quad (35)$$

for a fixed value of V_0 and d. This rough equality is a consequence of the fact that the factors kS_n and $N_{n,n}$ are not critically dependent on the modal characteristics. Because the transverse variation function $f(y_s)$ for surface waves is exponentially attenuated from the boundaries, it follows that, for a slit source, we should take $y_s=0$, if the ratio given by (35) is to be maximized. Thus, considering a guiding structure consisting of two parallel plane surfaces of surface reactance $120\pi p$ ohms, it follows from (5), (18), and (35) that

$$\frac{P}{P_m} \sim \exp[+kpd] \quad (36)$$

is valid if $p \ll 1$. Not surprisingly, this result indicates that a major portion of the power is being supplied mostly to the surface wave if kd is sufficiently large that kpd is greater than 3 or 4.

Lossy Structures

Up to this point, the discussion has been related to a lossless structure. The formal theory is, of course, valid for complex surface impedances Z_0 and Z_d but then the excited modes will be attenuated. In the present context, the interesting case is when the resistive parts of the surface impedances are small compared with the inductive parts. To this end, set

$$Z_0 = i\eta p(1 - i\delta_0) \quad \text{and} \quad Z_d = i\eta q(1 - i\delta_d)$$

where $\eta = k/\epsilon\omega \cong 120\pi$ and p, q, δ_0, and δ_d are all real but $\delta_0 \ll 1$ and $\delta_d \ll 1$.

These various formulas, developed above for the propagation constants of the surface waves and the waveguide modes, are still valid if p and q are replaced everywhere by $p(1-i\delta_0)$ and $q(1-i\delta_d)$. Thus, the horizontal wavenumber for the surface wave, in the case of negligible interaction, is given by

$$\begin{aligned}\lambda &= k[1 + p^2(1 - i\delta_0)^2]^{1/2} \\ &\cong k(1 + p^2)^{1/2} - ikp^2\delta_0(1 + p^2)^{-1/2}. \end{aligned} \tag{37}$$

Therefore, the attenuation rate is $kp^2\delta_0(1+p^2)^{-1/2}$ nepers per unit length.

For the waveguide modes, it is a simple matter to show from (27) that the attenuation rate is approximately $(1/d)(p\delta_0+q\delta_d)[1-(\pi m/kd)^2]^{-1/2}$ for a mode of order m. This result, of course, is only valid if p and q are also small, and it is required that the modes are not near cutoff.

The ratio R of the attenuation rate of a surface wave mode and a waveguide mode is an interesting parameter. Using the immediately foregoing results, it follows that

$$R \cong \frac{kdp^2\delta_0}{p\delta_0 + q\delta_d}\left[1 - \left(\frac{\pi m}{kd}\right)^2\right]^{1/2} \tag{38}$$

which indicates that a lossy upper boundary (i.e., $\delta_d > 0$) will attenuate the waveguide modes but will have a negligible effect on the attenuation of the surface wave on the lower boundary.

Equation (38) above is valid only if both p and q are small. The other interesting case is to take p and q to be sufficiently large that, for the waveguide modes, $|C_m| \ll p$ and q. Under this condition, (21) may be approximated by

$$C_m\left[ikd + \frac{1}{ip} + \frac{1}{iq} + \frac{\delta_0}{p} + \frac{\delta_d}{q}\right] = i\pi m \tag{39}$$

which gives an explicit formula for calculating C_m. The corresponding attenuation rate for a waveguide mode of order m is readily found to be given approximately by $k(\pi m/kd)^2(\delta_p/p+\delta_d/q)$. For this case, the ratio of the surface wave to the waveguide mode attenuation rate is found to be

$$R \cong \left(\frac{kd}{\pi m}\right)^2 \frac{p^3}{(1 + p^2)^{1/2}\left(1 + \frac{p}{q}\frac{\delta_d}{\delta_0}\right)} \tag{40}$$

where $m = 1, 2, 3, \cdots$.

This ratio R can be made small by choosing the upper boundary to be relatively lossy (i.e., $\delta_d \gg \delta_0$). However, one should be cautious in drawing too many conclusions from (40) because of the approximations used. For example, it is required that $kpd > 2$ or 3 in order to neglect the surface wave interaction with the upper boundary and the corresponding waveguide modes must be sufficiently near grazing that both p and $q \gg (\pi m/kd)$.

Conclusion

It is believed that some of the results given here have a bearing on the operation of devices which require that a surface wave be contained in an enclosed conductor. While the idea of shielding the surface from the external environment is interesting, considerable care should be taken to avoid the contaminating influences of the waveguide modes which will accompany the desired surface wave modes. In spite of this fundamental drawback, the subject warrants further attention.

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